CHAPTER 17 Indefinite Integrals

Section-A JEE Advanced/ IIT-JEE

A Fill in the Blanks

1. If $\int \frac{4e^x + 6e^{-x}}{9e^x - 4e^{-x}}\, dx = Ax + B \log(9e^{2x} - 4) + C$, then $A =$ ...., $B =$ .... and $C =$ .... *(1990 - 2 Marks)*

C MCQs with One Correct Answer

1. The value of the integral $\int \frac{\cos^3 x + \cos^5 x}{\sin^2 x + \sin^4 x}\, dx$ is *(1995S)*
   (a) $\sin x - 6\tan^{-1}(\sin x) + c$
   (b) $\sin x - 2(\sin x)^{-1} + c$
   (c) $\sin x - 2(\sin x)^{-1} - 6\tan^{-1}(\sin x) + c$
   (d) $\sin x - 2(\sin x)^{-1} + 5\tan^{-1}(\sin x) + c$

2. If $\int_{\sin x}^{1} t^2 f(t)\, dt = 1 - \sin x$, then $f\left(\frac{1}{\sqrt{3}}\right)$ is *(2005S)*
   (a) $\frac{1}{3}$ (b) $\frac{1}{\sqrt{3}}$
   (c) 3 (d) $\sqrt{3}$

3. $\int \frac{x^2 - 1}{x^3\sqrt{2x^4 - 2x^2 + 1}}\, dx =$ *(2006 - 3M, –1)*
   (a) $\frac{\sqrt{2x^4 - 2x^2 + 1}}{x^2} + c$ (b) $\frac{\sqrt{2x^4 - 2x^2 + 1}}{x^3} + c$
   (c) $\frac{\sqrt{2x^4 - 2x^2 + 1}}{x} + c$ (d) $\frac{\sqrt{2x^4 - 2x^2 + 1}}{2x^2} + c$

4. Let $I = \int \frac{e^x}{e^{4x} + e^{2x} + 1}\, dx$, $J = \int \frac{e^{-x}}{e^{-4x} + e^{-2x} + 1}\, dx$. Then, for an arbitrary constant C, the value of $J - I$ equals *(2008)*
   (a) $\frac{1}{2}\log\left(\frac{e^{4x} - e^{2x} + 1}{e^{4x} + e^{2x} + 1}\right) + C$ (b) $\frac{1}{2}\log\left(\frac{e^{2x} + e^{x} + 1}{e^{2x} - e^{x} + 1}\right) + C$
   (c) $\frac{1}{2}\log\left(\frac{e^{2x} - e^{x} + 1}{e^{2x} + e^{x} + 1}\right) + C$ (d) $\frac{1}{2}\log\left(\frac{e^{4x} + e^{2x} + 1}{e^{4x} - e^{2x} + 1}\right) + C$

5. The integral $\int \frac{\sec^2 x}{(\sec x + \tan x)^{\frac{9}{2}}}\, dx$ equals (for some arbitrary constant K) *(2012)*
   (a) $-\frac{1}{(\sec x + \tan x)^{\frac{11}{2}}}\left\{\frac{1}{11} - \frac{1}{7}(\sec x + \tan x)^2\right\} + K$
   (b) $\frac{1}{(\sec x + \tan x)^{\frac{11}{2}}}\left\{\frac{1}{11} - \frac{1}{7}(\sec x + \tan x)^2\right\} + K$
   (c) $-\frac{1}{(\sec x + \tan x)^{\frac{11}{2}}}\left\{\frac{1}{11} + \frac{1}{7}(\sec x + \tan x)^2\right\} + K$
   (d) $\frac{1}{(\sec x + \tan x)^{\frac{11}{2}}}\left\{\frac{1}{11} + \frac{1}{7}(\sec x + \tan x)^2\right\} + K$

E Subjective Problems

1. Evaluate $\int \frac{\sin x}{\sin x - \cos x}\, dx$ *(1978)*

2. Evaluate $\int \frac{x^2\, dx}{(a + bx)^2}$ *(1979)*

3. Evaluate $\int (e^{\log x} + \sin x) \cos x\, dx$. *(1981 - 2 Marks)*

4. Evaluate : $\int \frac{(x-1)e^x}{(x+1)^3}\, dx$ *(1983 - 2 Marks)*

5. Evaluate the following $\int \frac{dx}{x^2(x^4+1)^{3/4}}$ *(1984 - 2 Marks)*

6. Evaluate the following $\int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}dx$ *(1985 - 2½ Marks)*

7. Evaluate : $\int \left[\frac{(\cos 2x)^{1/2}}{\sin x}\right]dx$ *(1987 - 6 Marks)*

8. Evaluate $\int (\sqrt{\tan x} + \sqrt{\cot x})dx$ *(1989 - 3 Marks)*

9. Find the indefinite integral $\int \left(\frac{1}{\sqrt[3]{x}+\sqrt[4]{4}} + \frac{\ln(1+\sqrt[6]{x})}{\sqrt[3]{x}+\sqrt{x}}\right)dx$

(1992 - 4 Marks)

10. Find the indefinite integral $\int \cos 2\theta \ln\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)d\theta$

(1994 - 5 Marks)

11. Evaluate $\int \frac{(x+1)}{x(1+xe^x)^2}dx.$ *(1996 - 2 Marks)*

12. Integrate $\int \frac{x^3+3x+2}{\left(x^2+1\right)^2(x+1)}dx.$ *(1999 - 5 Marks)*

13. Evaluate $\int \sin^{-1}\left(\frac{2x+2}{\sqrt{4x^2+8x+13}}\right)dx$. *(2001 - 5 Marks)*

14. For any natural number m, evaluate

$$\int (x^{3m}+x^{2m}+x^m)(2x^{2m}+3x^m+6)^{1/m}dx, x>0.$$

(2002 - 5 Marks)

H Assertion & Reason Type Questions

1. Let $F(x)$ be an indefinite integral of $\sin^2 x$.

STATEMENT-1 : The function $F(x)$ satisfies $F(x+\pi)=F(x)$ for all real x. **because**

STATEMENT-2 : $\sin^2(x+\pi)=\sin^2 x$ for all real x.

(2007 -3 marks)

(a) Statement-1 is True, statement-2 is True; Statement-2 is a correct explanation for Statement-1.

(b) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1

(c) Statement-1 is True, Statement-2 is False

(d) Statement-1 is False, Statement-2 is True.

Section-B JEE Main / AIEEE

1. If $\int \frac{\sin x}{\sin(x-\alpha)}dx = Ax + B\log\sin(x-\alpha), + C,$ then value of (A, B) is **[2004]**

(a) $(-\cos\alpha, \sin\alpha)$ (b) $(\cos\alpha, \sin\alpha)$

(c) $(-\sin\alpha, \cos\alpha)$ (d) $(\sin\alpha, \cos\alpha)$

2. $\int \frac{dx}{\cos x - \sin x}$ is equal to **[2004]**

(a) $\frac{1}{\sqrt{2}}\log\left|\tan\left(\frac{x}{2}+\frac{3\pi}{8}\right)\right|+C$

(b) $\frac{1}{\sqrt{2}}\log\left|\cot\left(\frac{x}{2}\right)\right|+C$

(c) $\frac{1}{\sqrt{2}}\log\left|\tan\left(\frac{x}{2}-\frac{3\pi}{8}\right)\right|+C$

(d) $\frac{1}{\sqrt{2}}\log\left|\tan\left(\frac{x}{2}-\frac{\pi}{8}\right)\right|+C$

3. $\int\left\{\frac{(\log x-1)}{1+(\log x)^2}\right\}^2 dx$ is equal to **[2005]**

(a) $\frac{\log x}{(\log x)^2+1}+C$ (b) $\frac{x}{x^2+1}+C$

(c) $\frac{xe^x}{1+x^2}+C$ (d) $\frac{x}{(\log x)^2+1}+C$

4. $\int \frac{dx}{\cos x + \sqrt{3}\sin x}$ equals **[2007]**

(a) $\log\tan\left(\frac{x}{2}+\frac{\pi}{12}\right)+C$

(b) $\log\tan\left(\frac{x}{2}-\frac{\pi}{12}\right)+C$

(c) $\frac{1}{2}\log\tan\left(\frac{x}{2}+\frac{\pi}{12}\right)+C$

(d) $\frac{1}{2}\log\tan\left(\frac{x}{2}-\frac{\pi}{12}\right)+C$

5. The value of $\sqrt{2}\int\frac{\sin x dx}{\sin\left(x-\frac{\pi}{4}\right)}$ is **[2008]**

(a) $x+\log\left|\cos\left(x-\frac{\pi}{4}\right)\right|+c$

(b) $x-\log\left|\sin\left(x-\frac{\pi}{4}\right)\right|+c$

(c) $x+\log\left|\sin\left(x-\frac{\pi}{4}\right)\right|+c$

(d) $x-\log\left|\cos\left(x-\frac{\pi}{4}\right)\right|+c$

6. If the $\int\frac{5\tan x}{\tan x-2}dx = x + a\ln|\sin x - 2\cos x| + k$, then a is equal to : **[2012]**

(a) –1 (b) –2

(c) 1 (d) 2

7. If $\int f(x)dx = \psi(x)$, then $\int x^5 f(x^3)dx$ is equal to **[JEE M 2013]**

(a) $\frac{1}{3}\left[x^3\psi(x^3)-\int x^2\psi(x^3)dx\right]+C$

(b) $\frac{1}{3}x^3\psi(x^3)-3\int x^3\psi(x^3)dx+C$

(c) $\frac{1}{3}x^3\psi(x^3)-\int x^2\psi(x^3)dx+C$

(d) $\frac{1}{3}\left[x^3\psi(x^3)-\int x^3\psi(x^3)dx\right]+C$

8. The integral $\int\left(1+x-\frac{1}{x}\right)e^{x+\frac{1}{x}}dx$ is equal to **[JEE M 2014]**

(a) $(x+1)e^{x+\frac{1}{x}}+c$ (b) $-xe^{x+\frac{1}{x}}+c$

(c) $(x-1)e^{x+\frac{1}{x}}+c$ (d) $xe^{x+\frac{1}{x}}+c$

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9. The integral $\int \frac{dx}{x^2(x^4+1)^{3/4}}$ equals : **[JEE M 2015]**

(a) $-(x^4+1)^{\frac{1}{4}}+c$

(b) $-\left(\frac{x^4+1}{x^4}\right)^{\frac{1}{4}}+c$

(c) $\left(\frac{x^4+1}{x^4}\right)^{\frac{1}{4}}+c$

(d) $(x^4+1)^{\frac{1}{4}}+c$

10. The integral $\int \frac{2x^{12}+5x^9}{\left(x^5+x^3+1\right)^3}dx$ is equal to : **[JEE M 2016]**

(a) $\frac{x^5}{2\left(x^5+x^3+1\right)^2}+C$

(b) $\frac{-x^{10}}{2\left(x^5+x^3+1\right)^2}+C$

(c) $\frac{-x^5}{\left(x^5+x^3+1\right)^2}+C$

(d) $\frac{x^{10}}{2\left(x^5+x^3+1\right)^2}+C$

where C is an arbitrary constant.

17 Indefinite Integrals

Section-A : JEE Advanced/ IIT-JEE

A **1.** $\frac{-3}{2}, \frac{35}{36}$, any real value

C **1.** (c) **2.** (c) **3.** (d) **4.** (c) **5.** (c)

E **1.** $\frac{1}{2}\log|\sin x-\cos x|+\frac{x}{2}+C$

2. $\frac{1}{b^3}\left[a+bx-2a\log|a+bx|-\frac{a^2}{a+bx}\right]+C$

3. $x\sin x+\cos x-\frac{1}{4}\cos 2x+C$

4. $\frac{e^x}{(x+1)^2}+C$

5. $-\left(1+\frac{1}{x^4}\right)^{\frac{1}{4}}+C$

6. $-2\sqrt{1-x}+\cos^{-1}\sqrt{x}+\sqrt{x}\sqrt{1-x}+C$

7. $\frac{1}{\sqrt{2}}\log\left[\frac{\sqrt{2}+\sqrt{1-\tan^2 x}}{\sqrt{2}-\sqrt{1-\tan^2 x}}\right]-\log(\cot x+\sqrt{\cot^2 x-1})+C$

8. $\sqrt{2}\tan^{-1}\left(\frac{\sqrt{\tan x}-\sqrt{\cot x}}{\sqrt{2}}\right)+c$

9. $\frac{3}{2}x^{2/3}-\frac{12}{7}x^{7/12}+2x^{1/2}-\frac{12}{5}x^{5/12}+3x^{1/3}-4x^{1/4}+6x^{1/6}-12x^{1/12}$

$+12\log|x^{1/2}+1|+6\left[\left\{\frac{(1+x^{1/6})^3}{3}-\frac{3}{2}(1+x^{1/6})^2+3(1+x^{1/6})\right\}\right.$

$\ln(1+x^{1/6})-\left\{\frac{(1+x^{1/6})^3}{9}-\frac{3}{4}(1+x^{1/6})^2+3(1+x^{1/6})\right\}$

$\left.+\frac{(\ln(1+x^{1/6}))^2}{2}\right]+C$

10. $\frac{\sin 2\theta}{2}\ln\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)-\frac{1}{2}\ln\sec 2\theta+C$

11. $\log\left(\frac{1+xe^x}{xe^x}\right)-\frac{1}{1+xe^x}+C$

12. $-\frac{1}{2}\log|x+1|+\frac{1}{4}\log(x^2+1)+\frac{3}{2}\tan^{-1}x+\frac{x}{1+x^2}+C$

13. $(x+1)\tan^{-1}\left(\frac{2x+2}{3}\right)-\frac{3}{4}\log(4x^2+8x+13)+C$

14. $\frac{1}{6}\frac{(2x^{3m}+3x^{2m}+6x^m)^{\frac{m+1}{m}}}{m+1}+C$

H **1.** (d)

Section-B : JEE Main/ AIEEE

1. (b) **2.** (a) **3.** (d) **4.** (c) **5.** (c) **6.** (d) **7.** (c)

8. (d) **9.** (b) **10.** (d)

Section-A JEE Advanced/ IIT-JEE

A. Fill in the Blanks

1. $\int \frac{4e^x + 6e^{-x}}{9e^x - 4e^{-x}} dx = Ax + B\ln(9e^{2x} - 4) + C$

$\Rightarrow \frac{d}{dx}[Ax + B\ln(9e^{2x} - 4) + C] = \frac{4e^x + 6e^{-x}}{9e^x - 4e^{-x}}$

$\Rightarrow A + \frac{18Be^x}{9e^x - 4e^{-x}} = \frac{4e^x + 6e^{-x}}{9e^x - 4e^{-x}}$

$\Rightarrow \frac{(9A + 18B)e^x - 4Ae^{-x}}{9e^x - 4e^{-x}} = \frac{4e^x + 6e^{-x}}{9e^x - 4e^{-x}}$

$\Rightarrow 9A + 18B = 4; -4A = 6 \Rightarrow A = \frac{-3}{2};$

$B = \left(4 + \frac{27}{2}\right)\frac{1}{18} = \frac{35}{36};$ C can have any real value.

C. MCQs with ONE Correct Answer

1. (c) Let $I = \int \frac{\cos^3 x + \cos^5 x}{\sin^2 x + \sin^4 x} dx = \int \frac{(\cos^2 x + \cos^4 x)\cos x}{\sin^2 x(1 + \sin^2 x)} dx$

$= \int \frac{[1 - \sin^2 x + (1 - \sin^2 x)^2]\cos x}{\sin^2 x(1 + \sin^2 x)} dx$

$= \int \frac{(2 - 3\sin^2 x + \sin^4 x)\cos x}{\sin^2 x(1 + \sin^2 x)} dx$

Put $\sin x = t \Rightarrow \cos x\, dx = dt$

$I = \int \frac{2 - 3t^2 + t^4}{t^4 + t^2} dt,\ I = \int \left(1 + \frac{2}{t^2} - \frac{6}{t^2 + 1}\right) dt$

$= t - \frac{2}{t} - 6\tan^{-1}(t) + C$

$= \sin x - 2(\sin x)^{-1} - 6\tan^{-1}(\sin x) + C$

2. (c) Given that $\int_{\sin x}^{1} t^2 f(t)\, dt = 1 - \sin x$

$\Rightarrow \frac{d}{dx}\int_{\sin x}^{1} t^2 f(t)\, dt = \frac{d}{dx}(1 - \sin x)$

$\Rightarrow -\sin^2 x f(\sin x).\cos x = -\cos x$

$\Rightarrow f(\sin x) = \frac{1}{\sin^2 x} \Rightarrow f(1/\sqrt{3}) = \frac{1}{(1/\sqrt{3})^2} = 3$

3. (d) $\int \frac{x^2 - 1}{x^3\sqrt{2x^4 - 2x^2 + 1}} dx = \int \frac{x^2 - 1}{x^5\sqrt{2 - \frac{2}{x^2} + \frac{1}{x^4}}} dx$

$= \frac{1}{4}\int \frac{\frac{4}{x^3} - \frac{4}{x^5}}{\sqrt{2 - \frac{2}{x^2} + \frac{1}{x^4}}} dx$

Put $2 - \frac{2}{x^2} + \frac{1}{x^4} = t \Rightarrow \left(\frac{4}{x^3} - \frac{4}{x^5}\right) dx = dt$

$= \frac{1}{4}\int \frac{dt}{\sqrt{t}} = \frac{2\sqrt{t}}{4} + C = \frac{\sqrt{2 - \frac{2}{x^2} + \frac{1}{x^4}}}{2} + C$

$= \frac{\sqrt{2x^4 - 2x^2 + 1}}{2x^2} + C$

4. (c) Given $I = \int \frac{e^x}{e^{4x} + e^{2x} + 1} dx,$

$J = \int \frac{e^{-x}}{e^{-4x} + e^{-2x} + 1} dx = \int \frac{e^{3x}}{e^{4x} + e^{2x} + 1} dx$

$\therefore$ $J - I = \int \frac{e^x(e^{2x} - 1)}{e^{4x} + e^{2x} + 1} dx$

Let $e^x = t \Rightarrow e^x dx = dt$

$\therefore$ $J - I = \int \frac{t^2 - 1}{t^4 + t^2 + 1} dt = \int \frac{1 - \frac{1}{t^2}}{t^2 + 1 + \frac{1}{t^2}} dt$

Let $t - \frac{1}{t} = u \Rightarrow \left(1 - \frac{1}{t^2}\right) dt = du$

$\therefore$ $J - I = \int \frac{du}{u^2 - 1} = \frac{1}{2}\log\left|\frac{u - 1}{u + 1}\right| + C$

$= \frac{1}{2}\log\left|\frac{\frac{t^2 + 1}{t} - 1}{\frac{t^2 + 1}{t} + 1}\right| + C = \frac{1}{2}\log\left|\frac{e^{2x} - e^x + 1}{e^{2x} + e^x + 1}\right| + C$

5. (c) $I = \int \frac{\sec^2 x}{(\sec x + \tan x)^{9/2}} dx$

Let $\sec x + \tan x = t \Rightarrow \sec x - \tan x = \frac{1}{t}$

$\Rightarrow \sec x = \frac{1}{2}\left(t + \frac{1}{t}\right)$

Also $\sec x(\sec x + \tan x)\, dx = dt$

$\Rightarrow \quad \sec x\, dx = \frac{dt}{t}$

$\therefore \quad I = \frac{1}{2}\int \frac{\left(t+\frac{1}{t}\right)dt}{t^{9/2}.t} = \frac{1}{2}\int\left(t^{-9/2}+t^{-13/2}\right)dt$

$= \frac{1}{2}\left[\frac{t^{-9/2+1}}{-\frac{9}{2}+1} + \frac{t^{-13/2+1}}{-\frac{13}{2}+1}\right] + K$

$= \frac{-1}{7}t^{-7/2} - \frac{1}{11}t^{-11/2} + K$

$= -\frac{1}{7t^{7/2}} - \frac{1}{11t^{11/2}} + K = -\frac{1}{t^{11/2}}\left(\frac{1}{11}+\frac{t^2}{7}\right) + K$

$= \frac{-1}{(\sec x + \tan x)^{11/2}}\left\{\frac{1}{11}+\frac{1}{7}(\sec x + \tan x)^2\right\} + K$

E. Subjective Problems

1. $I = \int \frac{\sin x}{\sin x - \cos x}dx = \frac{1}{2}\int \frac{2\sin x}{\sin x - \cos x}dx$

$= \frac{1}{2}\int \frac{\sin x + \cos x + \sin x - \cos x}{\sin x - \cos x}dx$

$= \frac{1}{2}\int \frac{\cos x + \sin x}{\sin x - \cos x}dx + \frac{1}{2}\int \frac{\sin x - \cos x}{\sin x - \cos x}dx$

$= \frac{1}{2}\log|\sin x - \cos x| + \frac{x}{2} + C$

2. Let $I = \int \frac{x^2 dx}{(a+bx)^2}$

Let $a + bx = t \Rightarrow x = \left(\frac{t-a}{b}\right) \Rightarrow dx = \frac{dt}{b}$

$\therefore \quad I = \frac{1}{b^3}\int \frac{(t-a)^2}{t^2}dt = \frac{1}{b^3}\int \frac{t^2 - 2at + a^2}{t^2}dt$

$= \frac{1}{b^3}\int\left(1 - \frac{2a}{t} + \frac{a^2}{t^2}\right)dt = \frac{1}{b^3}\left[t - 2a\log|t| - \frac{a^2}{t}\right] + C$

$= \frac{1}{b^3}\left[a + bx - 2a\log|a+bx| - \frac{a^2}{a+bx}\right] + C$

3. To evaluate $\int (e^{\log x} + \sin x)\cos x\, dx$

$= \int (x + \sin x)\cos x\, dx \qquad$ [Using $e^{\log x} = x$]

$= \int x\cos x + \frac{1}{2}\int \sin 2x\, dx$

$= x\sin x - \int \sin x\, dx + \frac{1}{2}\left(\frac{-\cos 2x}{2}\right)$

$= x\sin x + \cos x - \frac{1}{4}\cos 2x + C$

4. $I = \int \frac{(x-1)e^x}{(x+1)^3}dx = \int \frac{(x+1-2)e^x}{(x+1)^3}dx$

$= \int\left[\frac{1}{(x+1)^2} - \frac{2}{(x+1)^3}\right]e^x dx = \frac{e^x}{(x+1)^2} + C$

(Using $\int e^x(f(x) + f'(x)dx = e^x f(x))$)

5. Let $I = \int \frac{dx}{x^3x^2\left(1+\frac{1}{x^4}\right)^{3/4}}$

Put $1 + \frac{1}{x^4} = t \Rightarrow \frac{-4}{x^5}dx = dt \Rightarrow \frac{dx}{x^5} = -\frac{dt}{4}$

$\therefore \quad I = \int \frac{-dt}{4t^{3/4}} = \frac{-1}{4}\left(\frac{t^{-3/4+1}}{\frac{-3}{4}+1}\right) + C$

$= -t^{1/4} + C = -\left(1+\frac{1}{x^4}\right)^{1/4} + C$

6. $I = \int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}dx$

Put $x = \cos^2\theta \Rightarrow dx = -2\cos\theta\sin\theta\, d\theta$

$\therefore \quad I = -\int \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}.2\sin\theta\cos\theta\, d\theta$

$= -\int \frac{\sin\theta/2}{\cos\theta/2}.2.2\sin(\theta/2)\cos(\theta/2)\cos\theta\, d\theta$

$= -2\int (1-\cos\theta)\cos\theta\, d\theta$

$= -2\int (\cos\theta - \cos^2\theta)\, d\theta$

$= -2\int\left(\cos\theta - \frac{1+\cos 2\theta}{2}\right)d\theta$

$= -2\left[\sin\theta - \frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + C\right]$

$= -2\sqrt{1-x} + \left[\cos^{-1}\sqrt{x} + \sqrt{x}\sqrt{1-x}\right] + C$

[Using $\sin\theta = \sqrt{1-x}$]

$= -2\sqrt{1-x} + \cos^{-1}\sqrt{x} + \sqrt{x}\sqrt{1-x} + C$

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7. $I=\int\frac{\sqrt{\cos 2x}}{\sin x}dx=\int\frac{\sqrt{\cos^2 x-\sin^2 x}}{\sin^2 x}dx=\int\sqrt{\cot^2 x-1}\,dx$

Let $\cot x = \sec\theta \Rightarrow -\operatorname{cosec}^2 x\,dx = \sec\theta\tan\theta\,d\theta$

We get, $I=\int\sqrt{\sec^2\theta-1}.\frac{\sec\theta\tan\theta}{-(1+\sec^2\theta)}d\theta$

$$=-\int\frac{\sec\theta.\tan^2\theta}{1+\sec^2\theta}d\theta=-\int\frac{\sin^2\theta}{\cos\theta+\cos^3\theta}d\theta$$

$$=-\int\frac{1-\cos^2\theta}{\cos\theta(1+\cos^2\theta)}d\theta=-\int\frac{(1+\cos^2\theta)-2\cos^2\theta}{\cos\theta(1+\cos^2\theta)}d\theta$$

$$=-\int\sec\theta d\theta+2\int\frac{\cos\theta}{1+\cos^2\theta}d\theta$$

$$=-\log|\sec\theta+\tan\theta|+2\int\frac{\cos\theta}{2-\sin^2\theta}d\theta$$

$$=-\log|\sec\theta+\tan\theta|+2.\frac{1}{2\sqrt{2}}\log\left|\frac{\sqrt{2}+\sin\theta}{\sqrt{2}-\sin\theta}\right|+C$$

$$=-\log\left|\cot x+\sqrt{\cot^2 x-1}\right|+\frac{1}{\sqrt{2}}\log\left|\frac{\sqrt{2}+\sqrt{1-\tan^2 x}}{\sqrt{2}-\sqrt{1-\tan^2 x}}\right|+C$$

8. $I=\int(\sqrt{\tan x}+\sqrt{\cot x})dx=\int\frac{\sqrt{\sin x}}{\sqrt{\cos x}}+\frac{\sqrt{\cos x}}{\sqrt{\sin x}}dx$

$$=\int\frac{\sin x+\cos x}{\sqrt{\sin x\cos x}}dx=\sqrt{2}\int\frac{\sin x+\cos x}{\sqrt{\sin 2x}}$$

Put $\sin x-\cos x=t \Rightarrow (\cos x+\sin x)\,dx=dt$

also $(\sin x-\cos x)^2=t^2 \Rightarrow 1-\sin 2x=t^2$

$\Rightarrow \sin 2x=1-t^2$ $\qquad\because I=\sqrt{2}\int\frac{dt}{\sqrt{1-t^2}}$

$$=\sqrt{2}\sin^{-1}t+C=\sqrt{2}\sin^{-1}(\sin x-\cos x)+C$$

9. Let $I=\int\left(\frac{1}{\sqrt[3]{x}+\sqrt[4]{x}}+\frac{\ln(1+\sqrt[6]{x})}{\sqrt[3]{x}+\sqrt{x}}\right)dx$

$$=\int\frac{1}{\sqrt[3]{x}+\sqrt[4]{x}}dx+\int\frac{\ln(1+\sqrt[6]{x})}{\sqrt[3]{x}+\sqrt{x}}dx$$

$I=I_1+I_2$ ...(1)

where $I_1=\int\frac{1}{\sqrt[3]{x}+\sqrt[4]{x}}dx$

Let $x=y^{12}$ so that $dx=12y^{11}\,dy$

$\therefore\quad I_1=\int\frac{12y^{11}}{y^4+y^3}dy=12\int\frac{y^8}{1+y}dy$

$$=12\int\left(y^7-y^6+y^5-y^4+y^3-y^2+y-1+\frac{1}{y+1}\right)dy$$

$$=12\left[\frac{y^8}{8}-\frac{y^7}{7}+\frac{y^6}{6}-\frac{y^5}{5}+\frac{y^4}{4}-\frac{y^3}{3}+\frac{y^2}{2}-y+\log|y+1|\right]+C_1$$

$$=\frac{3}{2}x^{2/3}-\frac{12}{7}x^{7/12}+2x^{1/2}-\frac{12}{5}x^{5/12}+3x^{1/3}-4x^{1/4}+6x^{1/6}-12x^{1/12}+12\log|x^{1/12}+1|+C_1$$ ...(2)

$I_2=\int\frac{\ln(1+(x)^{1/6})}{(x)^{1/3}+(x)^{1/2}}dx$

Let $x=z^6$ so that $dx=6z^5\,dz$

$$=\int\frac{\ln(1+z)}{z^2+z^3}.6z^5dz=\int\frac{6z^3\ln(z+1)}{z+1}dz$$

Put $z+1=t \Rightarrow dz=dt$

$\therefore\quad I_2=\int\frac{6(t-1)^3\ln t}{t}dt=6\int(t^2-3t+3-\frac{1}{t})\ln t\,dt$

$$=6\left[\int(t^2-3t+3)\ln t\,dt-\int\frac{1}{t}\ln t\,dt\right]$$

$$=6\left[\left(\frac{t^3}{3}-\frac{3t^2}{2}+3t\right)\ln t-\int\left(\frac{t^3}{3}-\frac{3t^2}{2}+3t\right).\frac{1}{t}dt-\frac{(\ln t)^2}{2}\right]$$

$$=6\left[\left(\frac{t^3}{3}-\frac{3t^2}{2}+3t\right)\ln t-\int\left(\frac{t^2}{3}-\frac{3}{2}t+3\right)dt-\frac{(\ln t)^2}{2}\right]$$

$$=6\left[\left(\frac{t^3}{3}-\frac{3t^2}{2}+3t\right)\ln t-\left(\frac{t^3}{9}-\frac{3t^2}{4}+3t\right)-\frac{(\ln t)^2}{2}\right]+C_2$$

$$=6\left[\left\{\frac{(1+x^{1/6})^3}{3}-\frac{3}{2}(1+x^{1/6})^2+3(1+x^{1/6})\right\}\ln(1+x^{1/6})-\left\{\frac{(1+x^{1/6})^3}{9}-\frac{3}{4}(1+x^{1/6})^2+3(1+x^{1/6})\right\}+\frac{[\ln(1+x^{1/6})^2]}{2}\right]+C_2$$ ...(3)

Thus we get the value of I on substituting the values of I_1 and I_2 from (2) and (3) in equation (1).

10. Let $I=\int \cos 2\theta \ln\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)d\theta$

Now we observe that

$$\frac{d}{d\theta}\left\{\ln\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)\right\}$$

$$=\frac{d}{d\theta}[\ln(\cos\theta+\sin\theta)-\ln(\cos\theta-\sin\theta)]$$

$$=\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}-\frac{-\cos\theta-\sin\theta}{\cos\theta-\sin\theta}$$

$$=\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}+\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}$$

$$=\frac{\cos^2\theta+\sin^2\theta-2\sin\theta\cos\theta+\cos^2\theta+\sin^2\theta+2\sin\theta\cos\theta}{\cos^2\theta-\sin^2\theta}=\frac{2}{\cos 2\theta}$$

$\therefore$ Integrating I with respect to θ, by parts we get

$$I=\frac{\sin 2\theta}{2}\ln\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)-\int\frac{\sin 2\theta}{2}.\frac{2}{\cos 2\theta}d\theta$$

$$=\frac{\sin 2\theta}{2}\ln\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)-\int \tan 2\theta\, d\theta$$

$$=\frac{\sin 2\theta}{2}\ln\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)-\frac{1}{2}\ln\sec 2\theta+C$$

11. $I=\int\frac{(x+1)}{x(1+xe^x)^2}dx=\int\frac{e^x(x+1)}{xe^x(1+xe^x)^2}dx$

Put $1+xe^x=t \Rightarrow (xe^x+e^x)\,dx=dt$

$$\therefore\quad I=\int\frac{dt}{(t-1)t^2}=\int\left(\frac{1}{1-t}+\frac{1}{t}+\frac{1}{t^2}\right)dt$$

$$=-\log|1-t|+\log|t|-\frac{1}{t}+C$$

$$=\log\left|\frac{t}{1-t}\right|-\frac{1}{t}+C=\log\left|\frac{1+xe^x}{-xe^x}\right|-\frac{1}{1+xe^x}+C$$

$$=\log\left(\frac{1+xe^x}{xe^x}\right)-\frac{1}{1+xe^x}+C$$

12. $I=\int\frac{x^3+3x+2}{(x^2+1)^2(x+1)}dx.$

$$\frac{x^3+3x+2}{(x^2+1)^2(x+1)}=\frac{A}{x+1}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{(x^2+1)^2}$$

Comparing and solving, we get,

$$A=\frac{1}{2},B=\frac{1}{2},C=\frac{1}{2},D=0,E=2.$$

$$\therefore\quad I=-\frac{1}{2}\int\frac{1}{x+1}dx+\frac{1}{2}\int\frac{x+1}{x^2+1}dx+2\int\frac{dx}{(x^2+1)^2}$$

$$=-\frac{1}{2}\log|x+1|+\frac{1}{4}\log(x^2+1)+\frac{1}{2}\tan^{-1}x+2I_1+C$$

where $I_1=\int\frac{dx}{(x^2+1)^2}$, putting $x=\tan\theta$,

$$I_1=\int\frac{\sec^2\theta}{\sec^4\theta}d\theta=\int(\cos^2\theta)\,d\theta=\frac{1}{2}\int(1+\cos 2\theta)d\theta$$

$$=\frac{1}{2}(\theta+\frac{1}{2}\sin 2\theta)=\frac{1}{2}\tan^{-1}x+\frac{1}{4}.\frac{2x}{1+x^2}$$

$$\therefore\quad I=-\frac{1}{2}\log|x+1|+\frac{1}{4}\log(x^2+1)+\frac{3}{2}\tan^{-1}x+\frac{x}{1+x^2}+C$$

where C is constant of integration.

13. $I=\int\sin^{-1}\left(\frac{2x+2}{\sqrt{4x^2+8x+13}}\right)dx$

$$=\int\sin^{-1}\left[\frac{x+1}{\sqrt{x^2+2x+\frac{13}{4}}}\right]dx$$

$$=\int\sin^{-1}\left[\frac{x+1}{\sqrt{(x+1)^2)+(3/2)^2}}\right]dx$$

Put $x+1=3/2\tan\theta$, $dx=\frac{3}{2}\sec^2 d\theta$

$$\therefore\quad I=\int\sin^{-1}\left[\frac{\left(\frac{3}{2}\tan\theta\right)}{\sqrt{\frac{9}{4}\tan^2\theta+\frac{9}{4}}}\right]\frac{3}{2}\sec^2\theta\, d\theta$$

$$=\frac{3}{2}\int\sin^{-1}\left[\frac{\sin\theta}{\cos\theta}\frac{\cos\theta}{1}\right]\sec^2\theta d\theta$$

$$=\frac{3}{2}\int\theta\sec^2\theta\, d\theta=\frac{3}{2}\left[\theta\tan\theta-\int\tan\theta\, d\theta\right]$$

$$=\frac{3}{2}[\theta\tan\theta-\log|\sec\theta|]+C$$

$$I=\frac{3}{2}\left[\frac{2}{3}(x+1)\tan^{-1}\right]\left[\frac{2}{3}(x+1)\right]$$

$$-\log\sqrt{1+\frac{4}{9}(x+1)^2]}+C$$

$$= (x+1)\tan^{-1}\left(\frac{2x+2}{3}\right) - \frac{3}{4}\log(9+4x^2+8x+4) + \frac{3}{4}\log 9 + C$$

$$= (x+1)\tan^{-1}\left(\frac{2x+2}{3}\right) - \frac{3}{4}\log(4x^2+8x+13) + C$$

14. I $= \int (x^{3m}+x^{2m}+x^m)(2x^{2m}+3x^m+6)^{1/m}\,dx$

$$= \int (x^{3m}+x^{2m}+x^m)\left[\frac{2x^{3m}+3x^{2m}+6x^m}{x^m}\right]^{1/m} dx$$

$$= \int \left(\frac{x^{3m}+x^{2m}+x^m}{x}\right)(2x^{3m}+3x^{2m}+6x^m)^{1/m}\,dx$$

Put $2x^{3m}+3x^{2m}+6x^m = y$

$\therefore$ $I = \frac{1}{6m}\int y^{1/m}\,dy = \frac{1}{6m}\left(\frac{y^{1/m+1}}{1/m+1}\right) + C$

$$= \frac{1}{6}\left(\frac{y^{\frac{m+1}{m}}}{m+1}\right) + C$$

$$= \frac{1}{6}\frac{(2x^{3m}+3x^{2m}+6x^m)^{\frac{m+1}{m}}}{m+1} + C$$

H. Assertion & Reason Type Questions

1. **(d)** $F(x) = \int \sin^2 x\,dx = \frac{1}{2}\int (1-\cos 2x)dx$

$$= \frac{1}{4}(2x - \sin 2x) + C$$

Now, $F(x+\pi) = \frac{1}{4}(2x+2\pi-\sin(2x+2\pi)) + C$

$$= \frac{1}{4}[2x+2\pi-\sin 2x] + C \neq F(x)$$

$\therefore$ Statement – 1 is false.

Also $\sin^2(x+\pi) = \sin^2 x, \forall x \in R$

$\therefore$ Statement – 2 is true.

Section-B JEE Main/ AIEEE

1. **(b)** $\int \frac{\sin x}{\sin(x-\alpha)}dx = \int \frac{\sin(x-\alpha+\alpha)}{\sin(x-\alpha)}dx$

$= \int \frac{\sin(x-\alpha)\cos\alpha + \cos(x-\alpha)\sin\alpha}{\sin(x-\alpha)}dx$

$= \int \{\cos\alpha + \sin\alpha \cot(x-\alpha)\}dx$

$= (\cos\alpha)x + (\sin\alpha)\log\sin(x-\alpha) + C$

$\therefore A = \cos\alpha,\ B = \sin\alpha$

2. **(a)** $\int \frac{dx}{\cos x - \sin x} = \int \frac{dx}{\sqrt{2}\cos\left(x+\frac{\pi}{4}\right)}$

$= \frac{1}{\sqrt{2}}\int \sec\left(x+\frac{\pi}{4}\right)dx = \frac{1}{\sqrt{2}}\log\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}+\frac{\pi}{8}\right)\right| + C$

$\left[\because \int \sec x\,dx = \log\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|\right]$

$= \frac{1}{\sqrt{2}}\log\left|\tan\left(\frac{x}{2}+\frac{3\pi}{8}\right)\right| + C$

3. **(d)** $\int \frac{(\log x - 1)^2}{(1+(\log x)^2)^2}dx = \int \frac{1+(\log x)^2 - 2\log x}{\left[1+(\log x)^2\right]^2}dx$

$= \int\left[\frac{1}{(1+(\log x)^2)} - \frac{2\log x}{(1+(\log x)^2)^2}\right]dx$

$= \int\left[\frac{e^t}{1+t^2} - \frac{2t\,e^t}{(1+t^2)^2}\right]dt$ put $\log x = t \Rightarrow dx = e^t\,dt$

$= \int e^t\left[\frac{1}{1+t^2} - \frac{2t}{(1+t^2)^2}\right]dt$

[which is of the form $\int e^x\left(f(x)+f'(x)\right)dx$]

$= \frac{e^t}{1+t^2} + c = \frac{x}{1+(\log x)^2} + c$

4. **(c)** $I = \int \frac{dx}{\cos x + \sqrt{3}\sin x} \Rightarrow I = \int \frac{dx}{2\left[\frac{1}{2}\cos x + \frac{\sqrt{3}}{2}\sin x\right]}$

$= \frac{1}{2}\int \frac{dx}{\left[\sin\frac{\pi}{6}\cos x + \cos\frac{\pi}{6}\sin x\right]} = \frac{1}{2}\int \frac{dx}{\sin\left(x+\frac{\pi}{6}\right)}$

$\Rightarrow\quad I = \frac{1}{2}\int \operatorname{cosec}\left(x+\frac{\pi}{6}\right)dx$

But we know that $\int \operatorname{cosec} x\,dx = \log|(\tan x/2)| + C$

$\therefore\quad I = \frac{1}{2}\cdot\log\tan\left(\frac{x}{2}+\frac{\pi}{12}\right) + C$

5. **(c)** Let $I = \sqrt{2}\int \frac{\sin x dx}{\sin\left(x-\frac{\pi}{4}\right)}$ put $x - \frac{\pi}{4} = t$

$\Rightarrow\quad dx = dt \Rightarrow I = \sqrt{2}\int \frac{\sin\left(t+\frac{\pi}{4}\right)}{\sin t}dt$

$= \frac{\sqrt{2}}{\sqrt{2}}\int\left(\frac{\sin t + \cos t}{\sin t}\right)dt$

$\Rightarrow\quad I = \int(1+\cot t)dt = t + \log|\sin t| + c_1$

$= x - \frac{\pi}{4} + \log\left|\sin\left(x-\frac{\pi}{4}\right)\right| + c_1$

$= x + \log\left|\sin\left(x-\frac{\pi}{4}\right)\right| + c$ $\left(\text{where } c = c_1 - \frac{\pi}{4}\right)$

6. **(d)** $\int \frac{5\tan x}{\tan x - 2}dx = \int \frac{5\frac{\sin x}{\cos x}}{\frac{\sin x}{\cos x} - 2}dx$

$= \int\left(\frac{5\sin x}{\cos x} \times \frac{\cos x}{\sin x - 2\cos x}\right)dx$

$= \int \frac{5\sin x\,dx}{\sin x - 2\cos x}$

$= \int\left(\frac{4\sin x + \sin x + 2\cos x - 2\cos x}{\sin x - 2\cos x}\right)dx$

$= \int \frac{(\sin x - 2\cos x) + (4\sin x + 2\cos x)}{\sin x - 2\cos x}dx$

$= \int \frac{(\sin x - 2\cos x) + 2(\cos x + 2\sin x)}{(\sin x - 2\cos x)}dx$

$= \int \frac{\sin x - 2\cos x}{\sin x - 2\cos x}dx + 2\int\left(\frac{\cos x + 2\sin x}{\sin x - 2\cos x}\right)dx$

$= \int dx + 2\int \frac{\cos x + 2\sin x}{\sin x - 2\cos x}dx$

$= I_1 + I_2$ where $I_1 = \int dx$ and $I_2 = 2\int \frac{\cos x + 2\sin x}{\sin x - 2\cos x}dx$

put $\sin x - 2\cos x = t$

$\Rightarrow (\cos x + 2\sin x)\, dx = dt$

$\therefore I_2 = 2\int \frac{dt}{t} = 2\ln t + C = 2\ ln\ (\sin x - 2\cos x) + C$

Hence, $I_1 + I_2 = \int dx + 2\ln(\sin x - 2\cos x) + c$

$= x + 2ln\ |(\sin x - 2\cos x)| + k \Rightarrow a = 2$

7. **(c)** Let $\int f(x)dx = \psi(x)$

Let $I = \int x^5 f(x^3) dx$

put $x^3 = t \Rightarrow 3x^2 dx = dt$

$I = \frac{1}{3}\int 3 \cdot x^2 \cdot x^3 \cdot f(x^3) \cdot dx$

$= \frac{1}{3}\int tf(t)dt = \frac{1}{3}\left[t\int f(t)dt - \int f(t)dt\right]$

$= \frac{1}{3}\left[t\psi(t) - \int \psi(t)dt\right]$

$= \frac{1}{3}\left[x^3\psi(x^3) - 3\int x^2\psi(x^3)dx\right] + C$

$= \frac{1}{3}x^3\psi(x^3) - \int x^2\psi(x^3)dx + C$

8. **(d)** Let $I = \int\left(1 + x - \frac{1}{x}\right)e^{x+1/x}dx$

$= \int e^{x+1/x}dx + \int\left(x - \frac{1}{x}\right)e^{x+1/x}dx$

$= x.e^{x+1/x} - \int x\left(1 - \frac{1}{x^2}\right)e^{x+1/x}dx + \int\left(x - \frac{1}{x}\right)e^{x+1/x}dx$

$= x.e^{x+\frac{1}{x}} - \int\left(x - \frac{1}{x}\right)e^{x+1/x}dx + \int\left(x - \frac{1}{x}\right)e^{x+1/x}dx$

$= xe^{x+\frac{1}{x}} + C$

9. **(b)** $I = \int \frac{dx}{x^2(x^4+1)^{3/4}} = \int \frac{dx}{x^3(1+x^{-4})^{3/4}}$

Let $x^{-4} = y$

$\Rightarrow -4x^{-3}\, dx = dy \Rightarrow dx = \frac{-1}{4}x^3 dy$

$\therefore I = \frac{-1}{4}\int \frac{x^3 dy}{x^3(1+y)^{3/4}} = \frac{-1}{4}\int \frac{dy}{(1+y)^{3/4}}$

$= \frac{-1}{4} \times 4(1+y)^{1/4} = -(1+x^{-4})^{1/4} + C$

$= -\left(\frac{x^4+1}{x^4}\right)^{1/4} + C$

10. **(d)** $\int \frac{2x^{12}+5x^9}{(x^5+x^3+1)^3}dx$

Dividing by x^{15} in numerator and denominator

$\int \frac{\frac{2}{x^3} + \frac{5}{x^6}\, dx}{\left(1 + \frac{1}{x^2} + \frac{1}{x^5}\right)^3}$

Substitute $1 + \frac{1}{x^2} + \frac{1}{x^5} = t$

$\Rightarrow \left(\frac{-2}{x^3} - \frac{5}{x^6}\right)dx = dt$

$\Rightarrow \left(\frac{2}{x^3} + \frac{5}{x^6}\right)dx = -dt$

This gives,

$\int \frac{\frac{2}{x^3} + \frac{5}{x^6}\, dx}{\left(1 + \frac{1}{x^2} + \frac{1}{x^5}\right)^3} = \int \frac{-dt}{t^3} = \frac{1}{2t^2} + C$

$= \frac{1}{2\left(1 + \frac{1}{x^2} + \frac{1}{x^5}\right)^2} + C = \frac{x^{10}}{2(x^5+x^3+1)^2} + C$